

ARITHMETIC PROGRESSIONS IN MIDDLE  $\frac{1}{N}^{\text{th}}$  CANTOR SETS

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First to fix some notation. Let  $X \subset [0, 1]$  be the middle  $\frac{1}{N}^{\text{th}}$  Cantor set. That is  $X = \cap_{k=1}^{\infty} C_k$  where  $C_0 = [0, 1]$  and  $C_{k+1}$  is obtained by removing the middle  $\frac{1}{N}^{\text{th}}$  from each connected component of  $C_k$ . Notice  $C_k$  consists of  $2^k$  intervals of size  $(\frac{N-1}{2N})^k$ . The gaps between these intervals have size at least  $\frac{1}{N}(\frac{N-1}{2N})^{k-1}$ . Let  $a_1, \dots, a_r$  be numbers and  $X + a_r$  be considered modulo 1. For  $\delta > 0$  let  $X_\delta \supset X$  be the set obtained by deleting the middle  $N^{\text{th}}$  of size at least  $\delta$ . This is a finite union of intervals.

**Theorem 1.** *For any  $a_1, \dots, a_{\frac{N}{100 \log_2(N)}}$  we have that  $\cap_{i=1}^{\frac{N}{100 \log_2(N)}} X + a_i \neq \emptyset$ .*

That is, the middle  $\frac{1}{N}^{\text{th}}$  cantor set contains arithmetic progressions and in fact more general configurations of length proportional to  $\frac{N}{\log(N)}$ .

Broderick, Fishman and Simmons have subsequently proved this statement using variants of Schmidt's game [1, Theorem 2.1].

**Definition 2.** *We say an interval  $J$  of length  $\frac{1}{N^k}$  is  $k$ -good if*

$$J \cap \cap_{i=1}^{\frac{N}{100 \log_2(N)}} X_{\frac{1}{N^{k+1}}} + a_i$$

*contains  $\frac{N}{2}$  disjoint intervals of size  $\frac{1}{N^{k+1}}$ .*

We prove the Theorem by induction using the following Proposition:

**Proposition 3.** *If  $J$  is  $k$ -good then it contains a subinterval  $J'$  which is  $k+1$ -good.*

Notice that by compactness if  $J$  is a closed interval and

$$J \cap \cap_{i=1}^{\frac{N}{100 \log_2(N)}} X_{\frac{1}{N^{k+1}}} + a_i \neq \emptyset$$

for all  $k$  then

$$J \cap \cap_{i=1}^{\frac{N}{100 \log_2(N)}} X + a_i \neq \emptyset.$$

**Lemma 4.** *Let  $L > k$ . If  $J$  is an interval of size  $(\frac{N-1}{2N})^k$  and  $I_1, \dots, I_{2^{L-1}}$  be the intervals removed from  $C_{L-1}$  to obtain  $C_L$ . Then  $|\{r : I_r \cap J \neq \emptyset\}| \leq 2^{L-k-1}$ .*

*Proof.* This is maximized if  $J$  is a subinterval of  $X_{\frac{1}{N}(\frac{N-1}{2N})^{k-1}}$ . The estimate is achieved for those. To see that it is maximized for subintervals of  $X_{\frac{1}{N}(\frac{N-1}{2N})^{k-1}}$  let us consider a  $J$  with  $|J| = (\frac{N-1}{2N})^k$  so that the intersections with  $I_1, \dots, I_{2^{L-1}}$  are not contained in one subinterval of  $X_{\frac{1}{N}(\frac{N-1}{2N})^{k-1}}$ . So  $J$  is contained in  $U \cup G \cup V$  where  $U$  and  $V$  are subintervals of  $X_{\frac{1}{N}(\frac{N-1}{2N})^{k-1}}$  and  $G \subset ([0, 1] \setminus X_{\frac{1}{N}(\frac{N-1}{2N})^{k-1}})$  is the gap of size at least  $\frac{1}{N}(\frac{N-1}{2N})^{k-1}$  between them. We assume  $U$  is on the left of  $V$ . First notice no  $I_r$  is contained in  $G$ . Now if  $I_r \cap J \cap V \neq \emptyset$  then  $J = U + c$  where  $c - |G| \geq c - \frac{1}{N}(\frac{N-1}{2N})^{k-1} \geq d(I_r, q)$  where  $q$  is the left endpoint of  $V$ . Let  $p$

be the left endpoint of  $U$ . There exist  $I_L$  with  $d(I_L, p) = d(I_r, q)$ . Since  $|I_L| < |G|$  it follows that  $I_L \cap (U + c) = I_L \cap J = \emptyset$ . So by sliding  $U$  any new intersection with an  $I_j$  occurs only after a previous intersection with some  $I_r$  has been lost.  $\square$

**Corollary 5.** If  $J$  is any interval of size  $\frac{1}{N^k}$ , and  $I_1, \dots, I_r$  are the intervals of length exactly  $\frac{1}{N^k} \delta$  deleted to form  $X_{\delta \frac{1}{N^k}}$  then

$$|\{j : I_j \cap J \neq \emptyset\}| \leq 3 \cdot 2^{\log_{\frac{2N}{N-1}} \lceil \frac{1}{\delta} \rceil}.$$

*Proof.* Let  $p = \lceil \log_{\frac{2N}{N-1}} N^k \rceil$ .  $J$  contains at most parts of 3 subintervals of size  $(\frac{N-1}{2N})^p$ . Since there are at most  $\lceil \frac{1}{\delta} \rceil$  steps in the inductive process to form  $X$  between deleting intervals of size  $\frac{1}{N^k}$  and  $\delta \frac{1}{N^k}$ , The corollary follows by applying the lemma.  $\square$

*Proof of Proposition.* Consider the subintervals of  $J \cap_{i=1}^{\frac{N}{100 \log_2(N)}} X_{\frac{1}{N^{k+1}}} + a_i$  of size  $\frac{1}{N^{k+2}}$ . By the assumption that  $J$  is  $k$ -good we have at  $\frac{N^2}{2}$  disjoint intervals organized into  $\frac{N}{2}$  blocks of  $N$  consecutive intervals. (We may have other intervals too.) From  $X_{\frac{1}{N^{k+1}}}$  to  $X_{\frac{1}{N^{k+2}}}$  we can delete portions of at most

$$\begin{aligned} 3 \log_{\frac{2N}{N-1}}(N) 2^{\log_{\frac{2N}{N-1}}(N)} + 3N \log_{\frac{2N}{N-1}} N &\leq \\ 3 \cdot 2^{(\log_2 N)+1} \log_2 N + 3N \log_2 N &\leq 9N \log_2 N \end{aligned}$$

of them. This estimate follows because  $k$  intervals of total measure  $c$  can intersect at most  $2k + \delta^{-1}c$  disjoint intervals of size  $\delta$ . There are at most  $\log_{\frac{2N}{N-1}} N$  steps, and at each step we remove at most  $3 \cdot 2^{\log_{\frac{2N}{N-1}}(N)}$  intervals with total measure at most  $\frac{1}{N^k}$ .

We do this for each  $X + a_i$  and can delete portions of at most  $\frac{N^2}{20}$  intervals of size  $\frac{1}{N^{k+2}}$ . So by the pigeon hole principle one of the  $\frac{N^2}{2}$  blocks has at least half of its intervals. This is a  $k+1$ -good subinterval of  $J$ .  $\square$

**Remark 6.** The techniques of this note are a little robust and imply the existence of configurations for bilipshitz images of the middle  $\frac{1}{N}$  cantor set where the bilipshitz constant is not too large depending on  $N$ . It is natural to ask if there exists  $N$  so that the image of the middle  $\frac{1}{N}$  cantor set under any bilipshitz map contains 3 term arithmetic progressions.

**Question 1.** Is the bound found in this note on the order of the correct one? Is it possible to find arithmetic progressions say of order  $N$ ?

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## REFERENCES

- [1] Broderick, R; Fishman, L; Simmons, D, Quantitative results using variants of Schmidt's game: Dimension bounds, arithmetic progressions and more. *Preprint*.

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